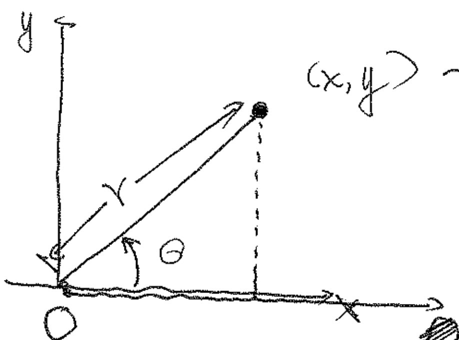
# §10.3 Polar Coordinates

Goal: In 10.1, 10.2, we introduced ONE new variable $t$ to rewrite the Cartesian equation, which gives us the parametric equations/curves. In 10.3, 10.4, we are going to introduce TWO new variables $(r, \theta)$ to re-study the given Cartesian equation. The new system is called POLAR COORDINATES

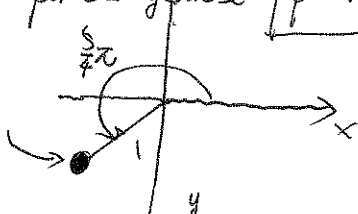
$(x, y) \sim (r, \theta)$.




$$\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \end{cases} \longleftrightarrow \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x}. \end{cases}$$

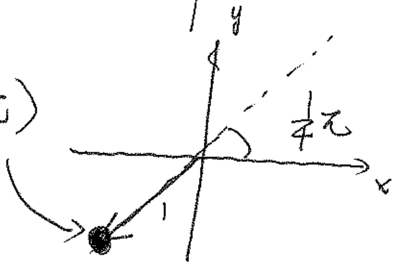
e.g. 1. Plot the points whose $\boxed{\text{polar coordinates}}$ are given

$(1, \frac{5}{4}\pi)$



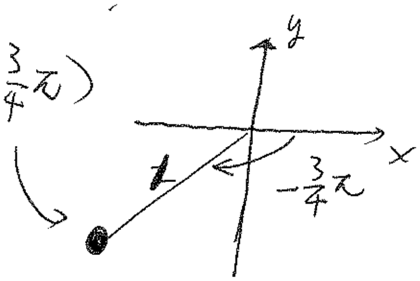

$(x, y)$ coordinates are $\begin{cases} x = 1 \cdot \cos\frac{5}{4}\pi = -\frac{\sqrt{2}}{2} \\ y = 1 \cdot \sin\frac{5}{4}\pi = -\frac{\sqrt{2}}{2} \end{cases}$

$(-1, \frac{1}{4}\pi)$



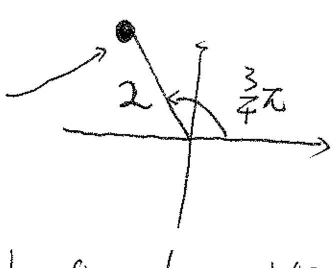

Remark: if $r < 0$, the point moves in the opposite direction in $\theta$ direction

$(1, -\frac{3}{4}\pi)$




Remark: if $\theta < 0$, the $\theta$-axis is moving clockwise.

$(2, \frac{3}{4}\pi)$




x, y coordinates are $x = 2 \cdot \cos\frac{3}{4}\pi = -\sqrt{2}$

$y = 2 \sin\frac{3}{4}\pi = \sqrt{2}$.

Notice that the first three points given in (different) polar coordinates are the same

- ☆ Polar Curves: The equation $r = f(\theta)$ is called a polar equation, the graph of this equation is called Polar Curves.

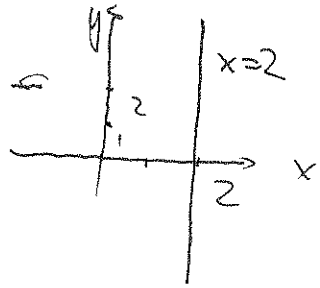
eg2. (Final 14). Consider the curve in polar coordinates: $r = 2\sec(\theta)$

Give ~~the~~ an equation for the curve in Cartesian $(x, y)$-coordinates. (What is the curve?)

sln: $\boxed{\text{Hint: } \sec\theta = \frac{1}{\cos\theta}}$, $r = 2\sec\theta = \frac{2}{\cos\theta} \iff r \cdot \cos\theta = 2$.

Recall $x = r\cos\theta = 2$

is a vertical line



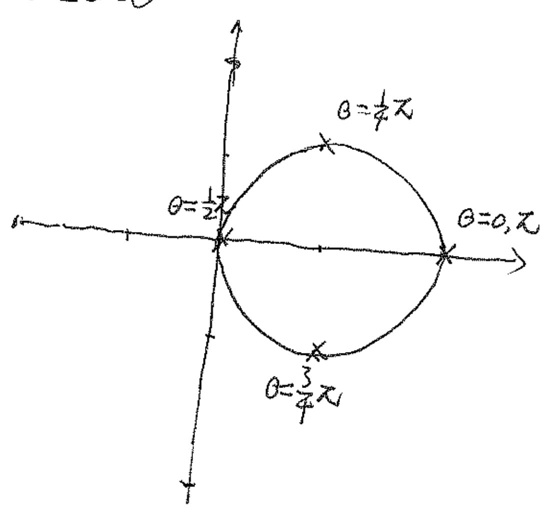

☆ The central examples are $r = a + b\cdot\cos\theta$ and $r = a + b\sin\theta$

for certain constants $a, b$.

eg.3 (a) Sketch the curve with polar equation $r = 2\cos\theta$

(Very important)

| | | $x = r\cos\theta$ | $y = r\sin\theta$ |
|---|---|---|---|
| $\theta = 0$, | $r = 2$ | 2 | 0 |
| $\theta = \frac{1}{4}\pi$, | $r = \sqrt{2}$ | 1 | 1 |
| $\theta = \frac{1}{2}\pi$ | $r = 0$ | 0 | 0 |
| $\theta = \frac{3}{4}\pi$, | $r = -\sqrt{2}$ | 1 | $-1$ |
| $\theta = \pi$ | $r = -2$ | 2 | 0 |




(b) Find the Cartesian equation for the curve:

Goal: Eliminate $r, \theta$ in $r = 2\cos\theta$ via $x = r\cos\theta$, $y = r\sin\theta$

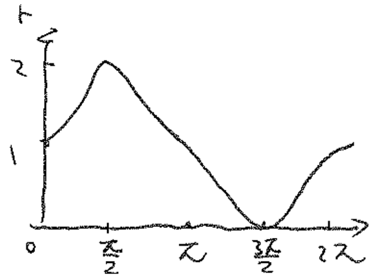
$\cos\theta = \frac{x}{r} \Rightarrow r = 2\cdot\frac{x}{r} \Rightarrow r^2 = 2x \xRightarrow{r^2 = x^2 + y^2} x^2 + y^2 = 2x$.

Remark: If we move on to convert the equation as $x^2 - 2x + 1 + y^2 = 1$

then $(x-1)^2 + y^2 = 1$, which is an equation of a circle centered at $(1, 0)$ with radius 1.

☆ e.g. 4 (very important) Sketch the curve $r = 1 + \sin\theta$

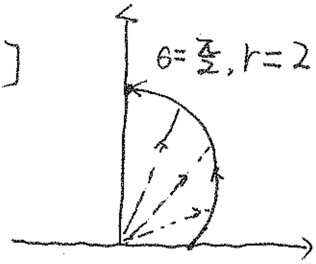
Hint:




$r = 1 + \sin\theta$

Remark: this is the graph in $(r, \theta)$ system, which is the same as $y = 1 + \sin x$.
NOT what we want. We want to plot the relation in $(x, y)$ system.
via $x = r\cos\theta$, $y = r\sin\theta$

$\theta \in [0, \frac{\pi}{2}]$

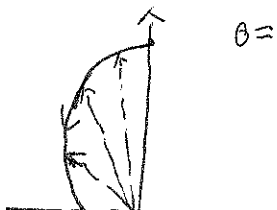


$\theta = \frac{\pi}{2}, r = 2, (x,y) = (0,2)$

$\theta = 0, r = 1$
$(x,y) = (1,0)$

$\theta$ from $0$ to $\frac{\pi}{2}$, $r = 1 + \sin\theta$ is increasing
(radius $r$ is getting longer and longer)

$\theta \in [\frac{\pi}{2}, \pi]$

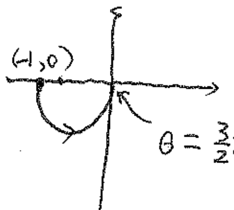


$\theta = \frac{\pi}{2}, r = 2, (x,y) = (0,2)$

$\theta = \pi, r = 1, (x,y) = (-1,0)$

$\theta$ from $[\frac{\pi}{2}, \pi]$, $r = 1 + \sin\theta$ is decreasing
($r$ is getting shorter and shorter)

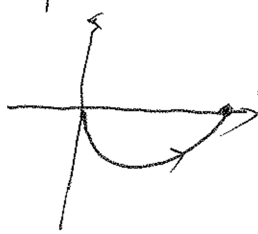
$\theta \in [\pi, \frac{3}{2}\pi]$




$\theta = \frac{3}{2}\pi, r = 0, (x,y) = (0,0)$

$\theta: \pi \to \frac{3\pi}{2}$. $r = 1 + \sin\theta$ decreasing

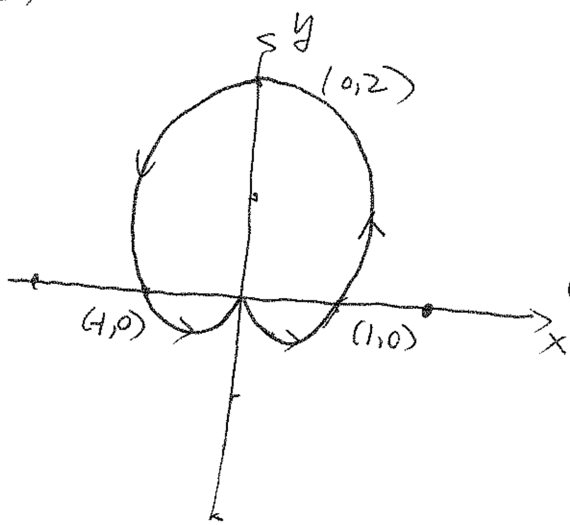
$\theta \in [\frac{3\pi}{2}, 2\pi]$



$\theta = 2\pi, r = 1, (x,y) = (1,0)$

$\theta: \frac{3\pi}{2} \to 2\pi$,
$r = 1 + \sin\theta$ increasing

The complete picture:
$r = 1 + \sin\theta$



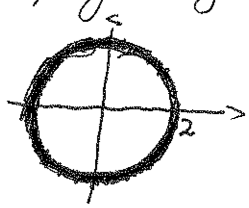

(cardioid)
(heart-shaped curve)

- Plane region (Circular sector) given by polar coordinates.

eg.5. Sketch the curves/regions given by the following polar (in)equalities.

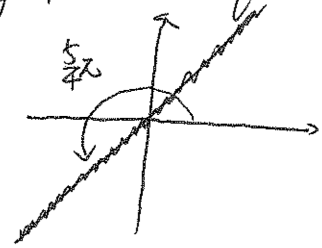
- $r=2$

(whole circle) $x^2+y^2=2^2$



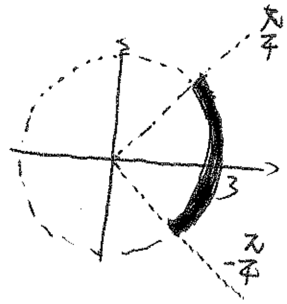

- $\theta=\frac{5}{4}\pi$

(line: $y=x$)



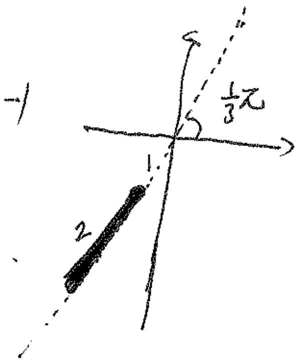

- $r=3$, $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$

quarter-circle of $x^2+y^2=3^2$




- $\theta=\frac{1}{3}\pi$, $-2\le r\le-1$

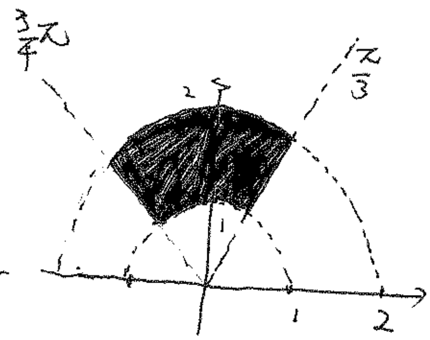
line segment of $y=\sqrt{3}x$.




- $1\le r\le 2$

$\frac{\pi}{3}\le\theta\le\frac{3}{4}\pi$

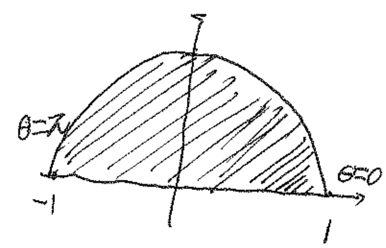
(empty) circular sector




- $0\le r\le 1$

$0\le\theta\le\pi$

(half-disk)




- Tangent to Polar curves:

eg.7. Find the tangent line (in x-y) to the polar curve $r=4\sin 2\theta$ at $\theta=\pi/4$

Remark: Give $r=f(\theta)$, $x=r\cdot\cos\theta$, $y=r\cdot\sin\theta$. We can eliminate $r$ to get parametric equations $x=f(\theta)\cdot\cos\theta$, $y=f(\theta)\sin\theta$, whose tangent line is studied in 1a1, 1a2.

Solution: $x=4\sin 2\theta\cdot\cos\theta$, $y=4\sin 2\theta\cdot\sin\theta$, at $\theta=\frac{\pi}{4}$, ~~$x=0$, $y=0$~~

$x=4\sin\frac{\pi}{2}\cdot\cos\frac{\pi}{4}=2\sqrt{2}$, $y=4\sin\frac{\pi}{2}\cdot\sin\frac{\pi}{4}=2\sqrt{2}$

$$\frac{dy}{d\theta}=4\cdot 2\cos 2\theta\cdot\sin\theta+4\sin 2\theta\cdot\cos\theta \overset{\theta=\pi/4}{=\!=} 8\cdot\cos\frac{\pi}{2}\sin\frac{\pi}{4}+4\sin\frac{\pi}{2}\cos\frac{\pi}{4}=2\sqrt{2}$$

$$\frac{dx}{d\theta}=4\cdot 2\cos 2\theta\cdot\cos\theta-4\sin 2\theta\sin\theta \overset{\theta=\pi/4}{=\!=} 8\cdot\cos\frac{\pi}{2}\cos\frac{\pi}{4}-4\sin\frac{\pi}{2}\sin\frac{\pi}{4}=-2\sqrt{2}$$

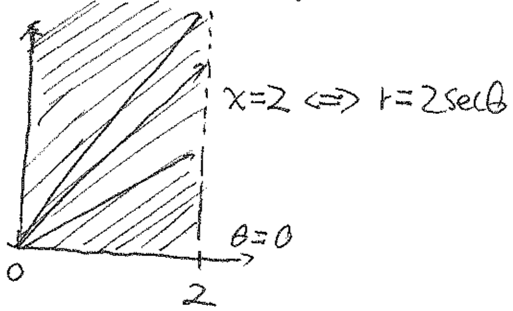
$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{+2\sqrt{2}}{-2\sqrt{2}}=-1.$$

tangent line: $\boxed{y=2\sqrt{2}-1\cdot(x-2\sqrt{2})}$

- Harder questions and Hints for ww.

eg.8 (related ww Part 1. ※12). Consider the polar curves in eg 2. $r = 2\sec\theta$.

① Sketch the region given by $0 \le r \le 2\sec\theta$, $0 \le \theta \le \frac{\pi}{2}$.




The region is $0 \le x \le 2$, $y \ge 0$

(it extends indefinitely in the y-direction)

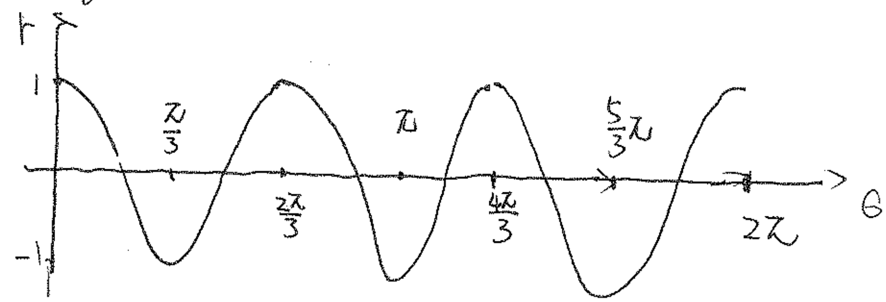
Remark: $\lim\limits_{\theta \to (\frac{\pi}{2})^-} \sec\theta = \lim\limits_{\theta \to (\frac{\pi}{2})^-} \frac{1}{\cos\theta} = \infty$.

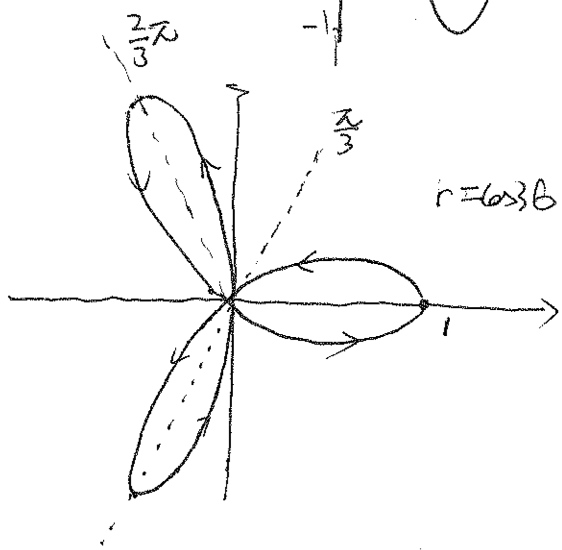
$\Rightarrow r \to +\infty$ as $\theta \to (\frac{\pi}{2})^-$.

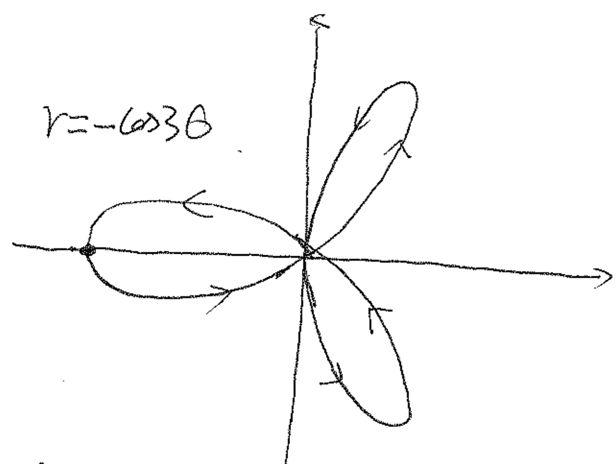
eg.9. Sketch the ~~region~~ curves $r = \cos 3\theta$ and $r = -\cos 3\theta$.

Cartesian $(r, \theta)$

$r = \cos 3\theta$




Remark: the same idea can be applied to

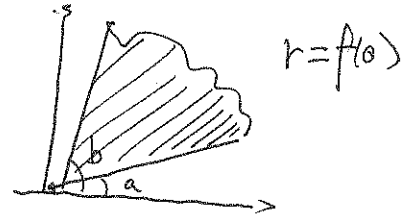
$r = \pm \sin 3\theta$, $r = \cos 2\theta$, $r = \sin 2\theta$, $r = \cos 4\theta$, $r = \sin 2\theta$.

The general curves are called k-rose.

# §10.4 Areas of Circular Section in polar coordinates.

**Formula:** Give a polar curve $r=f(\theta)$, $a\le\theta\le b$. The region ~~bounded~~ given by $0\le r\le f(\theta)$, $a\le\theta\le b$ has area:

$$\text{Area}=\int_a^b \frac{1}{2}\cdot[f(\theta)]^2\cdot d\theta.$$




Remark: In the formula sheet, the formula is given as $\text{Area}=\int_a^b \frac{1}{2}r(\theta)^2\,d\theta$, where $r(\theta)$ indicates the boundary curve $r=f(\theta)$.

eg1 (Final 14). Consider $r=2\sec\theta$. Find the area of the region bounded by this polar curve and the lines $\theta=0$, $\theta=\frac{\pi}{4}$.

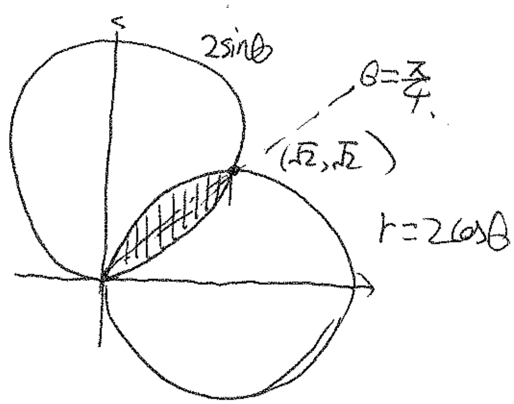
sln: $$\text{Area}=\int_0^{\frac{\pi}{4}}\frac{1}{2}\cdot[2\sec\theta]^2\,d\theta=\int_0^{\frac{\pi}{4}}2\sec^2\theta\cdot d\theta=2\tan\theta\Big|_0^{\frac{\pi}{4}}=2\tan\frac{\pi}{4}-2\tan 0$$
$$=\boxed{2}$$

★ eg2. Find the area shared by $r=2\sin\theta$ and $r=2\cos\theta$.

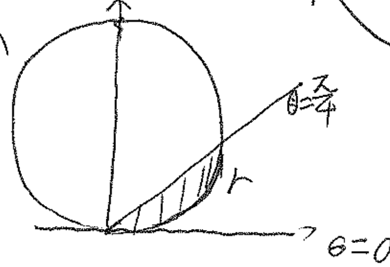
Step 1: intersection point.

$r=2\sin\theta=2\cos\theta\Rightarrow\sin\theta=\cos\theta$

$\Rightarrow\theta=\frac{\pi}{4}$. $(x,y)=(\sqrt{2},\sqrt{2})$




Step 2: It is enough to evaluate the area of the half shaded region

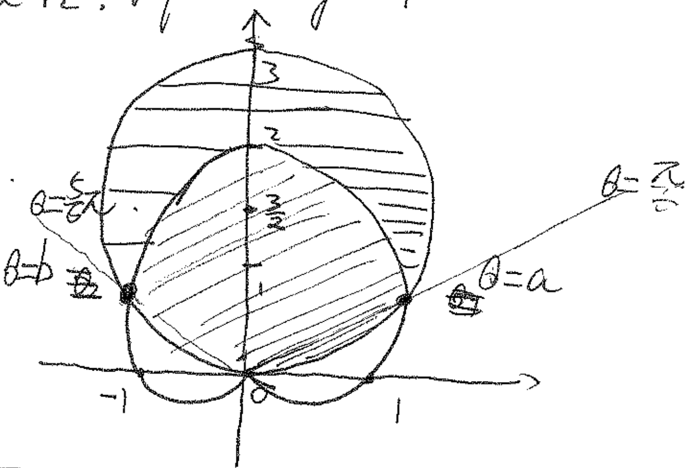



$$\frac{1}{2}\cdot\text{Area}=\int_0^{\frac{\pi}{4}}\frac{1}{2}(2\sin\theta)^2\,d\theta$$
$$=\int_0^{\frac{\pi}{4}}2\sin^2\theta\,d\theta$$
$$=\int_0^{\frac{\pi}{4}}1-\cos2\theta\,d\theta=\theta-\frac{1}{2}\sin2\theta\Big|_0^{\frac{\pi}{4}}=\frac{\pi}{4}-\frac{1}{2}$$

$$\boxed{\text{Area}=2\times\left[\frac{\pi}{4}-\frac{1}{2}\right]}$$

☆☆ e.g. 3. (very important) (textbook Page 690). Find the area of the region that lies inside the circle $r_1 = 3\sin\theta$ and outside the cardioid $r_2 = 1+\sin\theta$.

Remark: For "sketch the curves of $r_1$ and $r_2$", refer to eg.3 & 4 in §10.3.




Solution:

Step 1: Find the intersections, $\theta=a, \theta=b$, by setting $r_1 = r_2$, ie

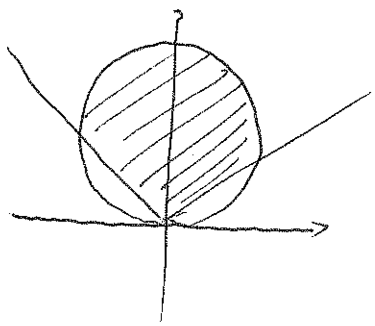
$$r_1 = 3\sin\theta = 1+\sin\theta = r_2$$

$$\Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2}.$$

$\boxed{\theta = \frac{\pi}{6}}$ (a) and $\boxed{\theta = \frac{5}{6}\pi}$ (b)

Step 2:

$r_1 = 3\sin\theta$



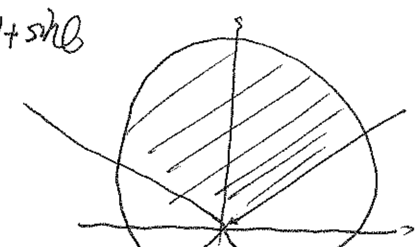
$$\text{Area 1} = \int_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} \frac{1}{2}\cdot r_1^2\, d\theta = \int_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} \frac{1}{2}(3\sin\theta)^2\, d\theta$$

$$= \int_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} \frac{9}{2}\cdot\frac{1-\cos 2\theta}{2}\, d\theta = \int_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} \frac{9}{4} - \frac{9}{4}\cos 2\theta\, d\theta$$

$$= \frac{9}{4}\theta - \frac{9}{8}\sin 2\theta \Big|_{\frac{1}{6}\pi}^{\frac{5}{6}\pi}$$

$$= \frac{3}{2}\pi - \frac{9}{8}\sqrt{3}$$

$r_2 = 1+\sin\theta$



$$\text{Area 2} = \int_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} \frac{1}{2} r_2^2\, d\theta = \int_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} \frac{1}{2}(1+\sin\theta)^2\, d\theta$$

$$= \int_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} \frac{1}{2}(1+2\sin\theta+\sin^2\theta)\, d\theta = \int_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} \frac{1}{2}\left(1+2\sin\theta+\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{3}{4}\theta - 2\cos\theta - \frac{1}{8}\sin 2\theta \Big|_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} = \frac{1}{2}\pi - \sqrt{3} - \frac{1}{8}\sqrt{3}$$

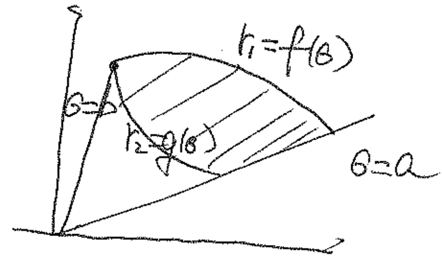
$$\text{Area} = \text{Area 1} - \text{Area 2} = \frac{3}{2}\pi - \frac{9}{8}\sqrt{3} - \left(\frac{1}{2}\pi - \frac{9}{8}\sqrt{3}\right) = \boxed{\pi}$$

Remark: You can directly compute

$$\text{Area} = \int_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} \frac{1}{2} r_1^2 - \frac{1}{2} r_2^2\, d\theta = \int_{\frac{1}{6}\pi}^{\frac{5}{6}\pi} \frac{1}{2}(3\sin\theta)^2 - \frac{1}{2}(1+\sin\theta)^2\, d\theta$$

Remark: In general, the region bounded by $g(\theta) \le r \le f(\theta)$, $a \le \theta \le b$

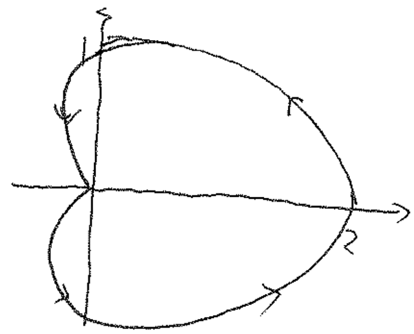
has area $= \displaystyle\int_a^b \frac{1}{2} f^2(\theta) - \frac{1}{2} \cdot g^2(\theta)\, d\theta$




• Arc-length in Polar coordinates: $(r, \theta)$. $x = r\cos\theta$, $y = r\cdot\sin\theta$

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, \cdot d\theta .$$

eg 4. Find the length of the cardioid $r = 1 + \cos\theta$, $\le \theta$




$\frac{dr}{d\theta} = -\sin\theta$

$$L = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1 + 2\cos\theta + \underline{\cos^2\theta + \sin^2\theta}}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2(1+\cos\theta)}\, d\theta \qquad \text{Hint: } \cos^2\frac{\theta}{2} = \frac{1+\cos 2\times\frac{\theta}{2}}{2} = \frac{1+\cos\theta}{2}$$

$$= \int_0^{2\pi} \sqrt{2 \cdot 2 \cdot \cos^2\frac{\theta}{2}}\, d\theta = \int_0^{2\pi} \left|2\cos\frac{\theta}{2}\right| d\theta = 2\int_0^{\pi} 2\cos\frac{\theta}{2}\, d\theta$$

$$= 4 \cdot 2 \cdot \sin\frac{\theta}{2} \cdot \Big|_0^{\pi} = 8$$

Remark: The area and arc-length formulas essentially follow from the parametric arc-length formula $L = \displaystyle\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, \cdot dt$ via u-sub or similar idea